

Generally covariant theories: the Noether obstruction for realizing certain space-time diffeomorphisms in phase space

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Abstract

Relying on known results of the Noether theory of symmetries extended to constrained systems, it is shown that there exists an obstruction that prevents certain tangent-space diffeomorphisms to be projectable to phase-space, for generally covariant theories. This main result throws new light on the old fact that the algebra of gauge generators in the phase space of General Relativity, or other generally covariant theories, only closes as a soft algebra and not a Lie algebra.

The deep relationship between these two issues is clarified. In particular, we see that the second one may be understood as a side effect of the procedure to solve the first. It is explicitly shown how the adoption of specific metric-dependent diffeomorphisms, as a way to achieve projectability, causes the algebra of gauge generators (constraints) in phase space not to be a Lie algebra —with structure constants— but a soft algebra —with structure *functions*.

1 Introduction

1.1 Diffeomorphisms in canonical general relativity

Diffeomorphisms are the gauge symmetries of general relativity (GR). However, it has been so far impossible to realize the complete Lie algebra of diffeomorphisms in the canonical formalism [1] (ADM) of GR. This -in principle- limitation of the canonical approach raises at least two immediate and relevant questions: a), one may wonder what the gauge group in canonical formalism of GR is, given that the diffeomorphism algebra is not properly realized, and b), one may ask for the reason that prevents this realization from being obtained.

On the other hand, and related to this fact, it is well known that the algebra of constraints in the phase space formulation of general relativity closes as a soft algebra, that is, with structure functions instead of structure constants, and not as a Lie algebra as one would have expected. This could also raise a new question, c), as to whether we realize any group structure at all for the gauge transformations in phase space.

As we said, these problems have been identified for a long time. As regards the questions related to the gauge group realization in phase space and its relationship to the diffeomorphism group (questions a) and c)), they have been answered some time ago, [2, 3, 4], and can be given a complete understanding in terms of the concept of a diffeomorphism-induced gauge group [5]. What we think still needs clarification is question b), as to why the canonical formalism of GR *can not* realize the Lie algebra of diffeomorphisms. We shall connect this question to that of the projectability of diffeomorphisms onto phase space.

In the approach taken in [2], and further pursued in [6, 7], the problem of realizing diffeomorphisms in phase space was properly addressed in the following way. The gauge generators in phase space are constructed with linear combinations of constraints containing arbitrary functions *and* their first time derivatives. The Poisson bracket of two of these generators, say $G[\xi_1], G[\xi_2]$, with ξ_1 and ξ_2 arbitrary functions of the space-time variables, should be of the form $G[\xi_3]$ for some ξ_3 , but the standard diffeomorphism rule $\xi_3^\mu = \xi_2^\nu \xi_{1,\nu}^\mu - \xi_1^\nu \xi_{2,\nu}^\mu$ can not be implemented by an equal-time commutator, as is the Poisson bracket, because $\dot{\xi}_3$, which appears in $G[\xi_3]$, will depend on the second time derivatives of ξ_1 and ξ_2 . This dependence can not be generated by the equal-time Poisson Bracket $\{G[\xi_1], G[\xi_2]\}$.

The main aim of this paper is to give a complementary and more structural explanation for the same fact from the general perspective of the Noether theory of symmetries, by pointing out where the obstruction resides that prevents certain diffeomorphisms to be projectable to phase space. This result will emerge as an

application of the Noether theory of symmetries extended to gauge theories¹, or more concretely, from the characterisation of the Noether conserved quantities in phase space.

A second aim of this paper is to show the deep connection between two issues, a) the one just mentioned concerning the Noether obstruction to obtain certain diffeomorphisms in phase space, and b) the well known fact that the algebra of gauge generators in the phase space of GR only closes as a soft algebra and not a Lie algebra. Our analysis will make it clear why this fact is inevitable and has its roots in the procedure adopted to circumvent the first problem, which is that of introducing field-dependent diffeomorphisms in order to achieve projectability. We may say that it seems very unlikely that there can be an alternative procedure, within the standard canonical formalism, to solve the first problem without causing the second one to appear. Even though we do not claim to solve the problem of the soft algebra realisation, we think that providing with a better understanding of its origin may open new ways to solve it or to prove that such a solution is not possible in the present framework.

To proceed, we will use the characterization, obtained in [8], of Noether symmetries (including both rigid and gauge symmetries) in the canonical formalism for gauge theories. So we will first quote the results we need in order to analyze the case of general relativity or, eventually, other gauge theories. Although we are interested in gauge field theories, we will use in this part the language of mechanics, which is sufficient for our purposes. A quick switch to the field theory language can be achieved by using DeWitt's [9] condensed notation².

In a series of papers, [5, 10, 11, 12, 13], the realization of the diffeomorphism-induced gauge group in phase space for several generally covariant theories has been studied, and the projectability issue of diffeomorphisms, which is examined at the level of the symmetry transformations, is addressed in detail. In contrast with this previous approach, the Noether theory for gauge systems will now allow us to directly link the projectability requirements of diffeomorphisms with specific properties of their corresponding Noether conserved quantities. This is one of the novelties of our approach.

It is worth mentioning another approach to addressing the problem of realizing the diffeomorphism algebra in phase space and circumventing the difficulties raised above. In the framework introduced in [14, 15]³, Isham and Kuchar enlarge

¹Here by gauge theories we mean theories, formulated through a variational principle, containing symmetries -gauge symmetries- that depend on arbitrary functions. This includes Yang-Mills theories, string theory, general relativity etc.

²As we shall see, spatial boundaries will not be relevant to our discussion, although they are indeed so when one takes into account that boundary terms can be needed for the action in order to get a correct formulation of the variational principle or the conservation of charges.

³See also the developments in [16] concerning the path integral formulation within this approach.

the phase space with a set of scalar fields that represent embedding space-time coordinates that have been promoted to the status of phase space variables. Remarkably it is then possible to obtain the diffeomorphism algebra in this augmented phase space. Let us also mention that in a phase space histories version [17] of canonical GR, the interpretation of two “types of time” makes it compatible, by defining a new Poisson bracket in the space of histories, the existence of both the diffeomorphism algebra structure and the constraints’ algebra structure.

2 Noether symmetries in gauge theories

Gauge theories present very specific features concerning the way the Noether theory of symmetries is implemented. Let us list the main ones. Consider, as our starting point a time-independent first-order Lagrangian $L(q, \dot{q})$ defined in tangent space TQ , that is, the tangent bundle of some configuration manifold Q . The -infinitesimal- Noether symmetries in tangent space we will consider are of the type $\delta^L q(q, \dot{q}; t)$. Gauge theories rely on singular -as opposed to regular- Lagrangians, that is, Lagrangians whose Hessian matrix with respect to the velocities (q stands, in a free index notation, for local coordinates in Q),

$$W_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}, \quad (1)$$

is not invertible.

Notice first that these -gauge- theories, having been defined through singular Lagrangians, allow for the possible existence of gauge -also called local- symmetries. These symmetries have the distinctive property of depending on arbitrary functions, and are deeply connected to certain identities, the Noether identities, which relate the Euler-Lagrange derivatives of the Lagrangian to their time -or space-time- derivatives to several orders. Clearly, these Noether identities are peculiar to singular Lagrangians.

A second feature resides in the special characteristics of the associated canonical formalism in phase space T^*Q . As Dirac showed [18, 19] in his pioneering work, dynamics can be still formulated in phase space but one must take into account two main novelties, absent in the regular case. First, the presence of constraints, that is, regions in phase space of low dimensionality identified as the only places where the equations of motion may have solutions; consistency of these constraints with the dynamics is always an important aspect to be analyzed and solved. And second, the presence of arbitrary functions in the dynamics as an explicit manifestation of the gauge phenomenon, which allows for the existence of *several* (in fact infinitely many) dynamical solutions, all starting with the *same* set of initial conditions. All these solutions must be considered as physically equivalent and are linked by gauge transformations.

A third feature concerns the very relation between the formalisms in velocity space and phase space. Since the singularity of the Hessian matrix makes the usual trade between velocities and momenta no longer possible, one is led to the issue of the *projectability* -or lack of it- of the structures from velocity space to phase space. This issue of projectability will be central in our presentation.

It is clear so far that there are many differences between the singular case -the one we are interested in- and the regular case (that is, when the map from velocity space to phase space is invertible). The extent to which these differences affect the formulation of Noether symmetries and conserved quantities is described below.

Either in the regular or in the singular case, a Noether conserved quantity $G^L(q, \dot{q}; t)$ and its associated infinitesimal transformation $\delta^L q(q, \dot{q}; t)$ are always linked by the basic relation

$$[L]_i \delta^L q^i + \frac{d^L}{dt}(G^L) = 0, \quad (2)$$

where $[L]_i$ stands for the Euler-Lagrange equations

$$[L]_i := \alpha_i - W_{is} \ddot{q}^s,$$

with

$$\alpha_i := -\frac{\partial^2 L}{\partial \dot{q}^i \partial q^s} \dot{q}^s + \frac{\partial L}{\partial q^i},$$

and the total time derivative is, in our case,

$$\frac{d^L}{dt} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i}. \quad (3)$$

2.1 The regular case

Now consider a regular theory. In this case, after the trading between velocities and momenta, we can write the Noether conserved quantity in phase space, $G^H(q, p; t)$ such that $G^L = \mathcal{FL}^*(G^H)$ (\mathcal{FL}^* is the pullback of the Legendre map from TQ to T^*Q , that is, $\mathcal{FL}^*(p) = \partial L / \partial \dot{q}$), and one question may easily come to mind: is there any characterization of G^H in phase space? the answer is yes: the canonical conserved quantity satisfies (H is the Hamiltonian)

$$\frac{\partial G^H}{\partial t} + \{G^H, H\} = 0, \quad (4)$$

if and only if its associated G^L satisfies (2) for some transformations $\delta^L q$. This transformation $\delta^L q$ can be written as

$$\delta^L q = \mathcal{FL}^*\{q, G^H\}, \quad (5)$$

explicitly showing that G^H is the generator of the canonical Noether transformation $\delta^H q = \{q, G^H\}$, which is the phase space version of $\delta^L q = \mathcal{FL}^*(\delta^H q)$. This classical result is remarkable: a single function, G^H , codifies all the information contained in the Noether symmetry $\delta^H q$.

Now the obvious question is: how can all these results that hold in a regular theory be translated, and to what extent, to the singular case? The precise answer is given in [8], relying on earlier work done in [20, 22].

2.2 The singular -gauge- case

In the singular case -singular Lagrangians defining gauge theories-, the conserved quantity G^L in (2) is still a projectable quantity [23], that is, there exists in phase space a function G^H such that $G^L = \mathcal{FL}^*(G^H)^4$. However, this function G^H is no longer unique because we can add to it an arbitrary linear combination of primary constraints. Also, in contrast with the regular case, and despite the existence of a conserved quantity in phase space, neither G^H nor any of its equivalent functions whose pull-back to tangent space is G^L , are guaranteed to generate the Noether transformation $\delta^L q$. By the same token, it is by no means guaranteed that $\delta^L q$ is a transformation projectable to phase space. These are the issues we will address next.

Before giving the complete results let us introduce some notation. The canonical Hamiltonian will be written H_c and it has the property that its pullback to TQ gives the Lagrangian energy, defined as $\dot{q} \partial L / \partial \dot{q} - L$. As we said, this canonical Hamiltonian has the ambiguity of the possible addition of some functions, the primary constraints, which we now introduce. The primary constraints, which will be denoted as ϕ_μ (with the index μ running the appropriate values) span a basis for the ideal of functions in T^*Q whose pullback to TQ under the Legendre map \mathcal{FL} vanishes, that is, $\mathcal{FL}^*(\phi_\mu) = 0$. Following Dirac, we can split the primary constraints between those that are first class, ϕ_{μ_0} , and the rest, second class, ϕ_{μ_1} , such that,

$$\{\phi_{\mu_0}, \phi_\mu\} = pc, \quad \det |\{\phi_{\mu_1}, \phi_{\nu_1}\}| \neq 0, \quad (6)$$

where $\{-, -\}$ is the Poisson Bracket structure and pc stands for a generic linear combination of the *primary* constraints. We will assume henceforth that the determinant in (6) will be different from zero everywhere in the surface of primary constraints.

Let us also mention that the vector field generating the dynamics (time evolution) in the canonical formalism for gauge theories (also called *constrained*

⁴An important example of this assertion is Dirac's canonical Hamiltonian itself, whose pull-back is the Lagrangian energy, the Noether conserved quantity associated with time translations.

systems in the Dirac's sense [18, 19]) is:

$$\mathbf{X}_H := \frac{\partial}{\partial t} + \{-, H_c\} + \lambda^\mu \{-, \phi_\mu\}, \quad (7)$$

where λ^μ are arbitrary functions of time. Consistency of the dynamics (7) with the primary constraints leads generally to new constraints and to new refinements of the dynamics (see [20]).

It is very convenient at this point to realize that the functions λ^μ , despite being, at this moment, arbitrary functions in phase space, admit an unambiguous determination in velocity space, namely, the functions $v^\mu(q, \dot{q})$ that satisfy the identity

$$\dot{q} \equiv \mathcal{F}L^*\{q, H_c\} + v^\mu(q, \dot{q})\mathcal{F}L^*\{q, \phi_\mu\}. \quad (8)$$

These functions v^μ are strictly non projectable to phase space. They form in fact [25] a basis for the functions that are not projectable to phase space. They will play an important role shortly.

It is proved in [8] that the phase space characterization of G^H for gauge theories is as follows:

Theorem 1 (García-Pons, 2000) *The pullback G^L –in TQ – of a function G^H in T^*Q satisfies (2) –for some δq^L – if and only if G^H satisfies*

$$\frac{\partial G^H}{\partial t} + \{G^H, H_c\}^* = sc + pc, \quad \{G^H, \phi_{\mu_0}\}^* = sc + pc, \quad (9)$$

where sc (pc) represents a generic combination of secondary (primary) constraints.

These secondary constraints are obtained by requiring the consistency of the dynamics with the primary constraints (essentially, by requiring tangency conditions of the evolutionary vector field with respect to the surface of primary constraints; this requirement may also determine some of the arbitrary functions λ^μ in (7). Search for constraints may continue with tertiary constraints, etc.). A basis –perhaps redundant, perhaps even void– of secondary constraints can be written as

$$\phi_{\mu_0}^1 := \{\phi_{\mu_0}, H_c\}. \quad (10)$$

The Dirac bracket in (9) is defined, at this level of the surface of primary constraints, by

$$\{A, B\}^* := \{A, B\} - \{A, \phi_{\mu_1}\} M^{\mu_1 \nu_1} \{\phi_{\nu_1}, B\},$$

where $M^{\nu_1 \mu_1}$ is the matrix inverse of the Poisson bracket matrix of the primary second-class constraints (see (10)).

One can notice that (9) is insensitive -as it must be- to the ambiguity inherent in the definition of G^H , that is, under the addition to a given G^H of linear combinations of primary constraints. We can rewrite (9) using a specific notation for the coefficients in the secondary constraints:

$$\begin{aligned}\frac{\partial G^H}{\partial t} + \{G^H, H_c\}^* &= A^{\mu_0} \phi_{\mu_0}^1 + pc, \\ \{G^H, \phi_{\mu_0}\}^* &= B_{\mu_0}^{\nu_0} \phi_{\nu_0}^1 + pc.\end{aligned}\tag{11}$$

The coefficients A^{μ_0} in (11) can be absorbed through a suitable choice of G^H among all the functions whose pullback is G^L ; specifically, under the change $G^H \rightarrow G^H - A^{\mu_0} \phi_{\mu_0}$ we make the new A^{μ_0} to vanish. A new change, suggested in [19] as the definition of a “starred” function,

$$G^H \rightarrow G^H - \{G^H, \phi_{\mu_1}\} M^{\mu_1 \nu_1} \phi_{\nu_1},$$

makes irrelevant the use of the Dirac bracket in favor of the usual Poisson bracket. Notice though that under these changes we have eliminated, partially at least, the ambiguity inherent in the definition of G^H , so the mathematical characterization for *the most general* G^H whose pullback is a Noether conserved quantity, G^L , in velocity space, is still (9) (or (11)). Notice finally that the coefficients $B_{\mu_0}^{\nu_0}$ remain invariant under the changes of G^H allowed by the addition of arbitrary linear combinations of the primary constraints. These quantities $B_{\mu_0}^{\nu_0}$ play a fundamental role in what follows.

2.3 Reconstructing the Noether transformation and finding the Noether obstruction in phase space

As we shall now see, these coefficients $B_{\mu_0}^{\nu_0}$ bear the full responsibility -if they do not vanish- for some Noether transformations not being projectable to phase space. Indeed one can prove [8] that

Theorem 2 (García-Pons, 2000) *The reconstruction of the Noether transformation⁵ $\delta^L q$, out of a function G^H satisfying (11) goes as follows:*

$$\delta^L q = \mathcal{FL}^*(\{q, G^H - A^{\mu_0} \phi_{\mu_0}\}^*) - v^{\mu_0} \mathcal{FL}^*(B_{\mu_0}^{\nu_0} \{q, \phi_{\nu_0}\}^*).\tag{12}$$

⁵There is a small ambiguity in $\delta^L q$, as can be seen in (2), because we can add to $\delta^L q$ an arbitrary antisymmetric linear combination of the Euler-Lagrange derivatives, whereas (2) keeps unchanged. Usually these additions introduce accelerations in the new $\delta^L q$, not allowed in our framework. Let us also point out that in some cases there can even be identically vanishing Noether conserved quantities associated with non-trivial gauge Noether transformations [24]. Examples are the relativistic free particle –without auxiliary variable– and the Nambu-Goto action for the string.

This result is the extension to gauge theories of the simpler relation (5), derived for the regular case.

It is clear that in (12) we completely identify the obstruction that may prevent the projectability of $\delta^L q$ to phase space, namely, the existence of the second term on the left hand side. Indeed, when any of the coefficients $B_{\mu_0}^{\nu_0}$ are different from zero, the functions v^{μ_0} , which are intrinsically non-projectable, prevent $\delta^L q$ from being projectable.

Summarizing, given a Noether conserved quantity in tangent space, G^L , we have a direct procedure to construct its associated Noether transformation and to check whether it is projectable to phase space: first write any function G^H in T^*Q such that its pullback is G^L , $G^L = \mathcal{FL}^*(G^H)$ (the existence of such a G^H is guaranteed, and in practice it is not difficult to find one). Next use the relations (9), compulsory for our G^H , to obtain the coefficients A^{μ_0} and $B_{\mu_0}^{\nu_0}$ that allow for the reconstruction of $\delta^L q$. When the B coefficients all vanish the transformation $\delta^L q$ will be projectable. Otherwise it will not be.

3 Application to Noether gauge symmetries: the Maxwell theory

The most important examples of gauge systems in theoretical physics are Yang-Mills theory, general relativity (GR) and string theory. These cases share some features that allow to treat them together with regard to the construction of their respective Noether gauge symmetries. In fact, for our purposes, the essence of Yang-Mills theory is already present in the Maxwell theory, which is the case we will consider henceforth. We will limit ourselves to comparisons between Maxwell theory and GR⁶.

Gauge transformations in Maxwell theory - $U(1)$ symmetry- and in general relativity -active space-time diffeomorphisms- share the fact that the infinitesimal gauge transformation depends on arbitrary functions (a scalar function Λ in electromagnetism, the components α^μ of a vector field $\alpha^\mu \partial_\mu$ in general relativity) of space-time *and* their first space-time derivatives. The particular fact that the first time derivative of the arbitrary functions is necessarily present in both cases has direct implications for the structure of constraints of these theories: there must be secondary constraints, in addition to the primary ones. This point will be clarified below. The similarities do not stop there: both theories exhibit only primary and secondary constraints, and all of them are first class.

⁶String theory in the Brink-DiVecchia-Howe-Deser-Zumino-Polyakov formalism can be essentially treated along the same lines.

Since all constraints are first class, the Dirac bracket coincides with the Poisson bracket, so in the cases we are considering, equations (11) will be read with the Poisson bracket. It is easy to show that *every* secondary constraint (remember that all constraints are first class) will provide us with a Noether gauge transformation. Indeed, in order to generate a transformation depending on an arbitrary function, say ϵ^{μ_0} , let us attach it to the secondary constraint: $\epsilon^{\mu_0}\phi_{\mu_0}^1$. We can just sum over μ_0 and thus describe the whole set of gauge transformations. This object, $G^H := \epsilon^{\mu_0}\phi_{\mu_0}^1$, satisfies (9) trivially. The first condition is satisfied because we know that there are no tertiary constraints in this case and the second one because all constraints are first class. So we can plug our G^H into (11) to compute the coefficients A^{μ_0} and $B_{\mu_0}^{\nu_0}$ in order to build the gauge transformations (12).

3.1 Maxwell theory

In the case of pure \mathcal{EM} , from the Lagrangian

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

we get the canonical Hamiltonian

$$H_c = \int d\mathbf{x} \left[\frac{1}{2}(\vec{\pi}^2 + \vec{B}^2) + \vec{\pi} \cdot \nabla A_0 \right],$$

and a primary constraint $\pi^0 \simeq 0$. Stability of this constraint under the Hamiltonian dynamics leads to the secondary constraint $\dot{\pi}^0 = \{\pi^0, H_c\} = \nabla \cdot \vec{\pi} \simeq 0$. Both constraints are first-class and no more constraints arise. According to the considerations above, it is immediate to write down the quantity G^H that satisfies (9),

$$G^H[\Lambda; t] = \int d^3x \Lambda(\mathbf{x}, t) \nabla \cdot \vec{\pi}(\mathbf{x}, t),$$

Λ being the arbitrary gauge function. One readily determines the quantities A in (11) and realizes that the quantities B vanish. Therefore the gauge transformation is projectable to phase space and canonically generated -through the Poisson bracket- by

$$G[\Lambda; t] = \int d^3x \left[-\dot{\Lambda}(\mathbf{x}, t)\pi^0(\mathbf{x}, t) + \Lambda(\mathbf{x}, t)\nabla \cdot \vec{\pi}(\mathbf{x}, t) \right].$$

The gauge transformation of the gauge field is then

$$\delta A_\mu = \{A_\mu, G\} = -\partial_\mu \Lambda,$$

which is the usual Noether $U(1)$ symmetry for the Lagrangian \mathcal{L}_M . Let us observe that a primary and a secondary constraint are necessary to ensure that the gauge field A_μ transforms covariantly.

4 General Relativity

One could obviously write four gauge transformations, one for each secondary constraint, following the procedure used for \mathcal{EM} . But that will not directly answer our question as to whether diffeomorphisms in tangent space are projectable to phase space. To this end we should rather start by constructing the conserved quantity G^L associated with diffeomorphisms in tangent space, obtaining its phase space version G^H and then checking whether the conditions of projectability –the vanishing of the coefficients $B_{\mu_0}^{\nu_0}$ in (11)– are met. The most efficient way to get G^L in general relativity for the pure gravity case is by making use of the doubly contracted Bianchi identities, which are the geometric version of the Noether identities. Let us see how it works.

Now we will use the language of field theory. Consider a Lagrangian density \mathcal{L} , with some fields⁷ ψ_A and having the Noether gauge symmetry

$$\delta\psi^A = \epsilon f^A + (\partial_\mu \epsilon) f^{A\mu}, \quad (13)$$

for an arbitrary function ϵ of space-time and for given functions f_A, f_A^μ , of the fields and their first space-time derivatives. Extension to higher space-time derivatives of the arbitrary function is straightforward but it is not needed here.

Use of the Noether condition (2), and the fact that the infinitesimal function ϵ is arbitrary, produces the Noether identity ($[\mathcal{L}]_A$ stands for the Euler-Lagrange derivatives of \mathcal{L}),

$$[\mathcal{L}]_A f^A = \partial_\mu ([\mathcal{L}]_A f^{A\mu}), \quad (14)$$

from which we can obtain

$$[\mathcal{L}]_A \delta\psi^A = \partial_\mu (\epsilon [\mathcal{L}]_A f^{A\mu}), \quad (15)$$

which identifies the conserved current as an object vanishing on shell, $J^\mu := \epsilon [\mathcal{L}]_A f^{A\mu}$. Notice that (13) and (14) are connected in both ways: one can either derive the Noether identity (14) from the gauge transformation (13) or vice-versa, construct the gauge transformation out of the Noether identity.

Let us apply these ideas to general relativity. The Einstein tensor density,

$$\mathcal{G}_{\mu\nu} := \sqrt{|g|} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}),$$

satisfies the Bianchi identities

$$\mathcal{G}_{;\nu}^{\mu\nu} = 0. \quad (16)$$

Now, in the case of pure gravity the Euler-Lagrange derivatives of the Einstein-Hilbert Lagrangian are just the components of the Einstein tensor density,

$$[\mathcal{L}_{EH}]^{\mu\nu} = \mathcal{G}^{\mu\nu},$$

⁷Unless stated otherwise, internal or space-time indices for each field will not be displayed.

and the content of (16) becomes that of a Noether identity. Indeed it can be equivalently expressed in the form of (15),

$$\mathcal{G}^{\mu\nu}\delta g_{\mu\nu} = \partial_\rho(2\epsilon^\lambda\mathcal{G}_\lambda{}^\rho), \quad (17)$$

with $\delta g_{\mu\nu} = \epsilon^\rho\partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu\epsilon^\rho + g_{\rho\nu}\partial_\mu\epsilon^\rho$, that is, the infinitesimal diffeomorphism -the Lie derivative- generated by the vector field $\epsilon^\rho\partial_\rho$. We recognize in (17) the Noether conserved current under the gauge symmetry of diffeomorphisms

$$J^\rho := 2\epsilon^\lambda\mathcal{G}_\lambda{}^\rho.$$

It is well known that there is an intrinsic ambiguity in the definition of the current J^ρ : a change of the type

$$J^\rho \rightarrow J^\rho + \partial_\mu A^{\mu\rho} \quad (18)$$

with any antisymmetric $A^{\mu\rho}$ leaves (17) invariant. The space integration of the time component J^0 is the putative Noether conserved charge $G^L = \int d^3x J^0$. Such conservation relies on the vanishing of the flux of the 3-vector J^i through the spatial boundary. Since J^0 vanishes on shell, it is clear that if there are to be any non-trivial conserved quantities at all, they must come about by way of the boundary terms afforded by (18). Here we see the relevance of the ambiguity in J^ρ , for it could be possible in some cases to adjust an $A^{\mu\rho}$ piece in such a way that the new G^L -which is the old one plus boundary terms- becomes a truly -and non-vanishing- conserved charge⁸. All the same these considerations are not important for our present purposes, because J^0 is already a constraint (in the Dirac sense) and hence G^L is trivially conserved -it vanishes on shell.

The components appearing in J^0 are the well known secondary constraints of pure GR. Using the Lapse and Shift functions, $N =: N^0$ and N^i respectively, and following the conventions of [21], they take the form

$$2\mathcal{G}_0{}^0 = N^\mu\mathcal{H}_\mu, \quad 2\mathcal{G}_i{}^0 = \mathcal{H}_i, \quad (19)$$

(here we have used a “covariant-like” notation, with indices $\mu = (0, i)$, just to express summation on the repeated indices, not to imply covariant behavior) where \mathcal{H}_0 is the Hamiltonian constraint and \mathcal{H}_i are the momentum constraints. Notice that to isolate \mathcal{H}_0 one essentially uses the unitary vector n^μ perpendicular to the equal-time foliation,

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right), \quad n_\mu = g_{\mu\nu}n^\nu = (-N, \vec{0}),$$

⁸The paramount example of this observation is the ADM energy.

in the following way

$$\begin{aligned} -N\mathcal{H}_0 &= -2\mathcal{G}_0^0 + 2N^i\mathcal{G}_i^0 = 2\mathcal{G}_0^0 n^0 n_0 + 2\mathcal{G}_i^0 n^i n_0 = 2\mathcal{G}_\mu^\nu n^\mu n_\nu \\ &= 2\mathcal{G}_{\mu\nu} n^\mu n^\nu = 2N\sqrt{\det g_{ij}} \left({}^{(3)}R + (K^i_i)^2 - K_{ij}K^{ij} \right), \end{aligned}$$

thus giving the standard definition [21] for the Hamiltonian constraint \mathcal{H}_0 , with K_{ij} being the extrinsic curvature of the equal-time surfaces.

Let us observe that

$$H_c := \int d^3x \, 2\mathcal{G}_0^0 = \int d^3x \, N^\mu \mathcal{H}_\mu, \quad (20)$$

is the canonical Hamiltonian (again, up to boundary terms that do not affect our discussion), that is, the Noether conserved charge when $\epsilon^\rho \partial_\rho = \partial_0$. Since the primary constraints of GR are just the momenta P_μ conjugate to the Lapse and Shift, note that the secondary constraints are

$$\{P_\mu, H_c\} = -\mathcal{H}_\mu.$$

Remark. In order to avoid any confusion with the standard literature, let us observe that, usually, many presentations of the canonical formalism for GR ignore the primary constraints P_μ . In such a case, the Lapse and Shift variables take on the role of Lagrange multipliers for a “Dirac” Hamiltonian (20), where the constraints \mathcal{H}_μ are taken as “primary” rather than “secondary”. Although this procedure is not incorrect, it fails to provide one with the full variables in phase space—in our case, *all* the components of the metric tensor and *all* their canonically conjugate momenta. In our procedure, instead, we consider that the true Dirac Hamiltonian is obtained by the addition to (20) of a combination of the primary constraints with arbitrary Lagrange multipliers, that is,

$$H_D := H_c + \lambda^\mu P_\mu. \quad (21)$$

Obviously the dynamics will impose $\dot{N}^\mu = \{N^\mu, H_D\} = \lambda^\mu$ and so we eventually recover, as we should, the arbitrary character of the Lapse and Shift. Notice, though, the advantage that we keep the phase space treatment complete, including Lapse and Shift and their canonical conjugates as ordinary variables, at every stage. Keeping the phase space treatment complete means that we do not lose the rules of transformation for the Lapse and Shift variables (or, equivalently, for the components $g^{\mu 0}$ of the metric). In this way, the connection between the gauge symmetries defined in phase space and the diffeomorphisms defined in tangent space can be made in very simple terms. Indeed the developments we undertake below are possible because we keep in the canonical formulation the entire configuration space of the Lagrangian formalism.

Now, the conserved charge associated with diffeomorphism invariance, generated by the infinitesimal vector field $\epsilon^\mu \partial_\mu$, is

$$G^L = \int d^3x J^0 = \int d^3x (\epsilon^0 N^\mu \mathcal{H}_\mu + \epsilon^i \mathcal{H}_i).$$

Taking into account the specific form of the Hamiltonian and momentum constraints, it is immediate to realize that they are trivially projectable to phase space. Therefore, with the understanding that we are now expressing our \mathcal{H}_μ in terms of the canonical variables, we can directly write down the canonical quantity (up to the addition of primary Hamiltonian constraints, whose pullback to tangent space identically vanishes) that satisfies (9):

$$G^H = \int d^3x J^0 = \int d^3x (\epsilon^0 N^\mu \mathcal{H}_\mu + \epsilon^i \mathcal{H}_i). \quad (22)$$

Now it is immediate to verify that the first piece in (22), that is, the conserved quantity $\int d^3x (\epsilon^0 N^\mu \mathcal{H}_\mu)$ associated with time diffeomorphisms, produces B pieces in the right side of (11), thus implying that the time diffeomorphisms are not projectable onto phase space. The obvious reason is the presence of the Lapse and Shift functions in this conserved quantity, which hit in the Poisson bracket with their conjugate variables, the primary constraints,

$$\left\{ \int d^3x \epsilon^0 N^\mu \mathcal{H}_\mu, P_\nu \right\} = \epsilon^0 \mathcal{H}_\nu,$$

thus identifying $B_\nu^\mu = -\delta_\nu^\mu \epsilon^0$.

Once the problem has been identified, we will focus on three issues. First, we will explain why this problem is unavoidable for generally covariant theories containing more than scalar fields; second, we will show the way out, which will of course make contact with standard formulations; and third, we will observe that this way out comes at a price: that the algebra of diffeomorphisms will not be realized as a Lie algebra, but as a “soft” algebra.

4.1 The reason for the non-projectability

We have seen that this non-projectability just happens. Now we will give a general argument explaining why this must be so.

The crucial observation is that the current in (15) depends only on the arbitrary function ϵ , and so does its time component J^0 (should the current in (15) depend on higher derivatives of ϵ , then $\delta\psi_A$ would not be a diffeomorphism transformation of a tensor field). Since its associated conserved quantity is directly

projectable to phase space, we can write $G^H = \int d^3x J^0$ as the quantity that fulfills (9). Then, use of (11) and (12) dictates the reconstruction of the diffeomorphism transformations out of G^H .

Now let us concentrate on time diffeomorphisms, and use the notation G_0^H for its associated conserved quantity. Observe that G_0^H only contains the arbitrary function $\epsilon^0 := \epsilon$, and *not* its time derivative $\dot{\epsilon}$ (space derivatives may be hidden behind spatial integration by parts). But to reconstruct a time diffeomorphism for the metric components, we need this time derivative (the only case when it is not needed is for the transformation of scalars). Where does this time derivative appear in (12)? It can only appear through the A terms. Thus we conclude that some A terms must be different from zero, which means that the theory we are considering, GR, must have secondary constraints.

This is the first step: we have deduced the existence of secondary constraints. Nothing new, of course, but the point is that this result has now been obtained from symmetry considerations alone.

Now comes the second step, which concerns the deep relationship between the conserved quantity G^H for time diffeomorphisms and Dirac's canonical Hamiltonian $H_c = \int d^3x \mathcal{H}_c$. We do not need to specify \mathcal{H}_c , but simply to realize that $J^0 = \epsilon \mathcal{H}_c$. The reason is that when ϵ is chosen to be an infinitesimal constant δt , then time diffeomorphisms become rigid time translations, and the conserved quantity associated with time translations is the canonical Hamiltonian⁹.

Now we are ready to get to the final point. Since there are secondary constraints in the theory, they must be produced through the Poisson bracket of the canonical Hamiltonian with the primary constraints. Schematically,

$$sc = \{pc, H_C\},$$

but then it is inescapable that the bracket $\{pc, G_0^H\}$, with $G_0^H = \int d^3x J^0 = \int d^3x \epsilon \mathcal{H}_c$, will develop also pieces with secondary constraints, thus producing B terms in (11) and hence leading to nonprojectability.

This proof of the nonprojectability of time diffeomorphisms to phase space applies to any generally covariant theory containing other than scalar fields.

⁹Let us stress, however, that despite the mathematical identification of time evolution with the symmetry of rigid time translations, both operations substantially differ [26] in their physical meaning: in the active view of diffeomorphism invariance –which is the one implicitly adopted throughout this paper–, time translations move a trajectory into another while *preserving the value of their time coordinate* (it is an equal-time operation) –or their space-time coordinates for field configurations, whereas the infinitesimal time evolution builds a single trajectory from time t to $t + \delta t$.

4.2 The way out

The nonprojectability of time diffeomorphisms introduces a potentially damaging problem for the formulation of GR in phase space, for it could imply that the full contents of diffeomorphism invariance can not be captured by the canonical formalism. That this is not true was shown in [5], where a discussion on the gauge group can be found. Here we will show a direct route to the same result.

Let us start by making a second look at our Noether conserved quantity for space-time diffeomorphisms, (22)

$$G^H = \int d^3x (\epsilon^0 N^\mu \mathcal{H}_\mu + \epsilon^i \mathcal{H}_i) = \int d^3x (\epsilon^0 N^0 \mathcal{H}_0 + (\epsilon^0 N^i + \epsilon^i) \mathcal{H}_i) . \quad (23)$$

Up to now, we have considered ϵ^μ as arbitrary functions of space-time. This means that we have not fully exploited the arbitrariness of these functions in order to build actions of the gauge group of GR. Using an arbitrary $\epsilon^\mu(x)$ (here x represents the space-time coordinates, $x = (t, \mathbf{x})$), we produce an infinitesimal action of the gauge group such that *all* field configurations (the space of field configurations is the natural arena for the action of the gauge group) undergo the *same* diffeomorphism. It is obvious that one can also consider the case where *different* field configurations can undergo *different* diffeomorphisms under the action of a single element of the gauge group¹⁰. This can be achieved by allowing ϵ^μ to have an arbitrary dependence not only on the space-time coordinates, but on the field configurations as well. With a specific selection of these dependences, which amounts to a change of basis for the infinitesimal generators of the gauge group, it turns out that we can achieve projectability.

In fact, let us make the functions ϵ^μ depend on the Lapse and Shift in such a way that any dependence on these variables disappears from (23). That is, we require that

$$\epsilon^0(x, N)N^0 = \xi^0(x), \quad \epsilon^0(x, N)N^i + \epsilon^i(x, N) = \xi^i(x),$$

for some functions $\xi^\mu(x)$ that only depend on the space-time coordinates. Inversion of these relations gives

$$\epsilon^\mu = n^\mu \xi^0 + \delta_i^\mu \xi^i, \quad (24)$$

where n^μ is the unitary vector orthogonal to the equal-time surfaces, introduced before.

Now G^H simplifies to

$$G^H = \int d^3x (\xi^0 \mathcal{H}_0 + \xi^i \mathcal{H}_i), \quad (25)$$

and it produces vanishing B functions in (11). We have achieved projectability, but it comes at a price.

¹⁰We refer to [5] for further considerations on the gauge group.

4.3 The price

The standard diffeomorphism algebra, for vectors $\vec{\epsilon}_1 = \epsilon_1^\mu(x)\partial_\mu$, $\vec{\epsilon}_2 = \epsilon_2^\mu(x)\partial_\mu$, is that of Lie derivatives: $\epsilon_3^\mu = \epsilon_2^\nu \epsilon_{1,\nu}^\mu - \epsilon_1^\nu \epsilon_{2,\nu}^\mu = [\vec{\epsilon}_2, \vec{\epsilon}_1]^\mu$. But this is only valid when the functions $\epsilon_1^\mu, \epsilon_2^\mu$ depend exclusively on the space-time coordinates. If $\epsilon_1^\mu, \epsilon_2^\mu$ are of the form (24), then ϵ_3^μ becomes

$$\epsilon_3^\mu = \epsilon_2^\nu \epsilon_{1,\nu}^\mu - \epsilon_1^\nu \epsilon_{2,\nu}^\mu + \xi_2^0 \mathcal{L}_{\vec{\epsilon}_1}(n^\mu) - \xi_1^0 \mathcal{L}_{\vec{\epsilon}_2}(n^\mu),$$

where $\mathcal{L}_{\vec{\epsilon}}(n^\mu)$ is the Lie derivative of the “vector” n^μ . But n^μ is not a true vector, for it is constructed algebraically out of the $g^{0\mu}$ components of the metric tensor. Indeed its transformation rules under the Lie derivative are [27]

$$\mathcal{L}_{\vec{\epsilon}}(n^\mu) = \mathcal{L}_{\vec{\epsilon}}^{(\text{naive})}(n^\mu) + N h^{\mu\nu} \partial_\nu \epsilon^0, \quad (26)$$

with $h^{\mu\nu} := g^{\mu\nu} + n^\mu n^\nu$ (we take the signature of g “mostly plus”), and

$$\mathcal{L}_{\vec{\epsilon}}^{(\text{naive})}(n^\mu) = [\vec{\epsilon}, \vec{n}]^\mu,$$

is the “naively” expected vector behavior for n^μ .

Now, expressing ϵ_3^μ as in (24), we obtain

$$\xi_3^\mu = \xi_2^i \xi_{1,i}^\mu - \xi_1^i \xi_{2,i}^\mu + h^{\mu\nu} (\xi_2^0 \xi_{1,\nu}^0 - \xi_1^0 \xi_{2,\nu}^0), \quad (27)$$

These equations already contain the Poisson bracket algebra of (25), that is,

$$\{G^H[\xi_1], G^H[\xi_2]\} = G^H[\xi_3], \quad (28)$$

out of which we can readily obtain the algebra of the first-class constraints \mathcal{H}_μ ,

$$\{\mathcal{H}_\mu, \mathcal{H}_\nu\} = C_{\mu\nu}^\sigma \mathcal{H}_\sigma,$$

with the structure functions $C_{\mu\nu}^\sigma$ being determined by (27) and (28).

The presence of the pieces $h^{\mu\nu}$ in (27) shows that the constraints \mathcal{H}_μ close with structure functions (depending on the field configurations) instead of structure “constants” (depending only on the space-time coordinates). This fact reflects the impossibility of realizing the Lie algebra of diffeomorphisms in phase space.

According to (12), in the case where the B coefficients vanish, the complete canonical generator of the gauge symmetries in phase space is $G^H - A^{\mu_0} \phi_{\mu_0}$, with the A coefficients determined by (11). In our case this gives (see [5] for a different derivation),

$$G[\vec{\xi}] = \int d^3x \left(P_\mu \dot{\xi}^\mu + (\mathcal{H}_\mu + N^\rho C_{\mu\rho}^\nu P_\nu) \xi^\mu \right), \quad (29)$$

Since in all the procedure we have not undertaken any gauge fixing, not even partially, we conclude that (29) is the general expression for the gauge generator in the entire phase space. As it was shown in [5], it describes locally (i.e., around the identity) the diffeomorphism-induced gauge group in phase space. In particular, its action on the configuration variables is that of a field-dependent diffeomorphism generated by the vector field (24).

Finally, let us remark that the deep reason why (27), and hence the coefficients $C_{\mu\nu}^\sigma$, exhibit field dependence is that, as we said before, the vector field n^μ in (24) is not a true vector field under diffeomorphisms. It turns out that the second term on the r.h.s. of (26), which reflects the deviation from the vector behavior, is directly responsible for the piece in (27) that carries the $h^{\mu\nu}$ dependence, and is the ultimate cause for the Poisson bracket algebra of the constraints \mathcal{H}_μ to close with structure functions and hence to form a soft algebra.

But, do we have still a group? Of course the gauge group, hugely larger than the diffeomorphisms group, is always a group. The fact that, for projectability reasons, we have chosen a basis of generators for the gauge group in phase space that close as a soft algebra, does not contradict any group law for the composition of our diffeomorphism-induced elements of the gauge group.

5 Conclusions

The canonical formalism for general relativity, relying on a $3 + 1$ decomposition, has a very specific, non-standard way of accommodating in phase space the full diffeomorphism invariance existing in tangent space. Being non-intrinsic, the $3 + 1$ decomposition is somewhat at odds with a generally covariant formalism, and difficulties arise for this reason. The non-projectability of some structures from tangent space to phase space is an example of such difficulties.

Nevertheless, in the case of diffeomorphisms, a kind of compromise is reached, and eventually we get some field-dependent diffeomorphisms that become projectable. A basis of infinitesimal projectable diffeomorphisms is thus obtained, and repeated iteration -that is, exponentiation- will provide us with the elements of the diffeomorphism-induced gauge group in phase space.

Let us list our main results.

1. In this paper we have identified the obstruction that prevents some infinitesimal diffeomorphisms of a generally covariant theory like GR from being realized in phase space. A novelty of our treatment is that this obstruction, which is an outcome of the Noether theory of symmetries extended

to gauge theories, is identified at the level of the characterisation of the Noether conserved quantities in phase space, (9).

2. We have also shown that this problem is *common to the canonical formulation of all generally covariant theories that contain fields other than scalars*. We give a complete explanation as to why this problem must be present. The essence of the argument is as follows. a) A generally covariant theory containing fields other than scalars must have first-class *secondary* constraints. b) These secondary constraints appear as the Poisson brackets of the canonical Hamiltonian with the first-class primary constraints. c) This canonical Hamiltonian is just the Noether conserved quantity associated with time translations -a particular case of an arbitrary time diffeomorphism. d) Therefore the conserved Noether quantity associated with arbitrary time diffeomorphisms will develop B terms in (11), thus making unavoidable the non-projectability of these time diffeomorphisms.
3. The adoption of field-dependent diffeomorphisms, in order to get projectability, appears as the natural and immediate way out within our formalism; we then recover standard formulas connecting these diffeomorphisms with the Noether transformations realized in phase space, which are generated -through the Poisson bracket- by specific combinations of the constraints of the theory, all first class. Let us mention that this method of regaining projectability becomes more involved when other gauge fields are present. For instance in Einstein-Yang-Mills theories, in addition to field-dependent diffeomorphisms one must also use [12] some field-dependent gauge rotations. Something similar happens when one uses the tetrad formalism [11] or the Ashtekar [28, 29, 30] complex formulation of canonical gravity [13, 10].
4. It is also shown that the resolution of the problem of projectability is unavoidably linked to the fact that the secondary constraints of GR -the so called Hamiltonian constraints- only close under structure functions. It is a consequence of our analysis that this is the price we must pay in order to solve the problem of projectability of diffeomorphisms. Going a little further we show that it is the failure of the “vector” field orthogonal to the equal-time surfaces to behave as a true vector under diffeomorphisms (this “vector” is constructed algebraically from the components of the metric tensor) that causes this closure under structure functions.
5. It is worth mentioning that using the doubly contracted Bianchi identities, (16), interpreted as the Noether identities, (14), in the pure gravity case of GR, gives a very efficient shortcut to obtain directly the Noether conserved currents associated with diffeomorphism invariance.

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